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A general solution for the dynamics of a generalized non-degenerate optical parametric down-conversion interaction by virtue of the Lewis–Riesenfeld invariant theory*

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Abstract

The dynamics of a generalized non-degenerate optical parametric down-conversion interaction whose Hamiltonian includes an arbitrary time-dependent driving part and a two-mode coupled part is studied by adopting the Lewis–Riesenfeld invariant theory. The closed formulae for the evolution of the quantum states and the evolution operators of the system are obtained. It is shown that various generalized squeezed states arise naturally in the process, and the two-mode squeezed effect is independent of the driving part. An explicitly analytical solution of the Schrödinger equation is further derived as the classical generalized force acting on each mode and the coupling of the two modes both have harmonic time dependences. This solution is found to be in agreement with previous research in special cases.

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1. Introduction

Over the past 30 years, the study of squeezed states of the electromagnetic field has attracted a great deal of attention because of their potential applications ranging from gravity wave detection [1] to reduction of quantum noises in optical communication systems [2]. Squeezed light has a lower value of noise dispersion in one quadrature component as compared with that of coherent light at the expense of larger noise in the complementary quadrature component [3].

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A powerful tool for generating squeezed states is optical parametric down-conversion. Down-conversion processes can be categorized as vacuum seeded, where only the high-frequency pump light beam is incident on the system, and bright seeded, where there is an additional incident low-frequency light beam. If the vacuum-seeded down-conversion takes place in a travelling wave system (no optical feedback), it is known as optical parametric fluorescence; if there is optical feedback (i.e. the material is inside a cavity), then the process is known as optical parametric oscillation. If the bright-seeded down-conversion occurs in an optical feedback system, it is known as optical parametric amplification. The quantum mechanical descriptions of a parametric down-conversion process in a degenerate parametric amplifier have been studied in detail [4–7]. Similar two-mode states were considered in various works [8–25]. The two-mode squeezed state, generated from a parametric down-conversion process with the Hamiltonian being [8, 10]

$$\hat{H}(t) = \sum_{j=1}^{2} \omega_j \hat{a}_j^+ \hat{a}_j + G + \mathrm{i}g\left\{\hat{a}_1\hat{a}_2 \exp[\mathrm{i}(\omega t + \phi)] - \hat{a}_1^+\hat{a}_2^+ \exp[-\mathrm{i}(\omega t + \phi)]\right\}, \tag{1.1}$$

has its idler-mode photon and signal-mode photon entangled with each other in the frequency domain, i.e. the correlation between idler mode and signal mode gives rise to two-mode squeezing [9]. On the other hand, it has been shown that the forced quantum oscillator subjected to a transient 'classical' driving force can generate coherent states [12]. In particular, the dynamics of a parametric amplifier whose Hamiltonian is composed of two forced quantum oscillators plus a parametric down-conversion interaction was studied very recently by virtue of the entangled state representation [9], where the solutions of the Schrödinger equation are derived, of which the simplest solution is a squeezed coherent state. However, in [9], only a special case is considered, i.e. the driving and pump parameters in the Hamiltonian were confined. In this work we shall study the dynamics of arbitrary time-dependent generalized parametric down-conversion by virtue of the Lewis–Riesenfeld invariant theory. In section 2 the dynamical system is defined in detail. In section 3 the closed formulae for the time evolution of the quantum states and the evolution operators of the system are obtained by selecting proper Hermitian invariant operator. In section 4, the analytical and explicit solutions of the Schrödinger equation are derived in detail when the classical generalized force acts on each mode and the coupling of the two modes have harmonic time dependences.

2. The dynamical system

The Hamiltonian of the time-dependent system in this study is (in natural units $\hbar = c = 1$):

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \tag{2.1a}$$

$$\hat{H}_0 = \sum_{j=1}^{2} \Omega_j^0(t)\hat{a}_j^+\hat{a}_j + G(t) \tag{2.1b}$$

$$\hat{H}_1 = \sum_{j=1}^{2} G_j(t)\left[\hat{a}_j^+ \exp(\mathrm{i}\varphi_j(t)) + \hat{a}_j \exp(-\mathrm{i}\varphi_j(t))\right] \tag{2.1c}$$

$$\hat{H}_2 = G_{12}(t)\left[\hat{a}_1^+\hat{a}_2^+ \exp(\mathrm{i}\phi(t)) + \hat{a}_1\hat{a}_2 \exp(-\mathrm{i}\phi(t))\right] \tag{2.1d}$$

where $\hat{a}_j$ and $\hat{a}_j^+$ are the annihilation and creation operations for the mode j ($j = 1, 2$; j is assigned the same values in what follows), respectively, and satisfy the following commutation relations,

$$\left[\hat{a}_j, \hat{a}_k^+\right] = \delta_{jk}, \qquad [\hat{a}_j, \hat{a}_k] = \left[\hat{a}_j^+, \hat{a}_k^+\right] = 0 \qquad (j, k = 1, 2) \tag{2.2}$$

where $\Omega_j^0(t)$, $G(t)$, $G_{12}(t)$, $\phi(t)$, $G_j(t)$ and $\varphi_j(t)$ are arbitrary real functions of time; $\hat{H}_0$ is the free Hamiltonian for the two-mode field; $\hat{H}_1$ is referred to as the driving part, $G_j(t)\exp(\mathrm{i}\varphi_j(t))$ can be regarded as a classical generalized driving force acting on the mode j [13]; $\hat{H}_2$ represents the two-mode coupled part, $G_{12}(t)$ and $\phi(t)$ are arbitrary pump coupling parameters. Generally speaking, when all parameters are arbitrarily time-dependent functions, the Hamiltonian (2.1) may describe a generalized parametric down-conversion process. Hereinafter, we will derive the closed solution of the Schrödinger equation for this general situation. When the classical generalized force acts on each mode and the coupling of the two modes both have harmonic time dependences, the analytical methods are at hand, i.e. we consider the harmonic time-dependent driving parameters

$$G_j(t) = G_j \cos\omega_j t, \qquad \varphi_j(t) = 0 \tag{2.3}$$

and the harmonic time-dependent pump coupling parameters

$$G_{12}(t)\exp[\mathrm{i}\phi(t)] = \frac{\kappa\varepsilon}{2}\exp\left[\mathrm{i}\left(\frac{\pi}{2} - 2\Omega t - \Phi\right)\right]. \tag{2.4}$$

Here G_j and ω_j are the time-independent amplitude and frequency of the driving force, respectively; κ describes an effective macroscopic non-linear coupling strength, ε is the constant pump amplitude (assumed to be undepleted), Φ is the phase difference between the pump field and the total phase of the two squeezed modes [10], 2Ω is the pump mode frequency. We suppose Ω_i^0 is the cavity mode frequency, and define

$$\Delta\Omega = 2\Omega - \Omega_1^0 - \Omega_2^0. \tag{2.5}$$

When $G_j = 0$ and the frequencies Ω_1^0 and Ω_2^0 sum up to the classical pump frequency 2Ω, i.e. $\Delta\Omega = 0$, we have parametric resonance. In this case the Schrödinger equation is straightforward to solve [11]. When $G_j(t) \neq 0$, $G_{12}(t) \neq 0$ and $\Delta\Omega \neq 0$, Hamiltonian (2.1) describes the detuned non-degenerate parametric down-conversion process acted by a classical generalized driving force inside a cavity, two down-conversion photons with frequencies Ω_1 and Ω_2, which are fixed by the phase-matching conditions and satisfy

$$2\Omega = \Omega_1 + \Omega_2 \tag{2.6}$$

are assumed to leave the cavity with a direction parallel to the pump. However, the cavity frequencies Ω_1^0 and Ω_2^0 might be slightly deviated from Ω_1 and Ω_2 due to uncontrollable broadening mechanisms (e.g. mechanical and thermal vibrations of the cavity mirrors which charge the cavity length). Certainly, the cavity is used in order to avoid the frequency broadening of its output signal and idler field (due to parameter fluorescence) and the spatial dispersion (light cone) [10]. In some particular cases, our model can return to those studied in previous researches [8–11].

3. A closed solution of the Schrödinger equation for the generalized parametric down-conversion

The time evolution of the quantum states follows the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle. \tag{3.1}$$

A Hermitian operator $\hat{I}(t)$ is called invariant if it satisfies

$$\mathrm{i}\frac{\partial}{\partial t}\hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = 0. \tag{3.2}$$

Because the two photons are always created together in a parametric down-conversion interaction, we can construct the Hermitian invariant by using unitary transformation of Hermitian operator $\hat{I}_0 = \hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2$, namely

$$\hat{I}(t) = \hat{D}_1(z_1(t))\hat{D}_2(z_2(t))\hat{S}(\xi(t))\hat{I}_0\hat{S}^+(\xi(t))\hat{D}_2^+(z_2(t))\hat{D}_1^+(z_1(t)). \quad (3.3)$$

Here $\hat{D}_j(z_j(t))$ is the displacement operator for mode j, defined by

$$\hat{D}_j(z_j(t)) = \exp\left[z_j(t)\hat{a}_j^+ - z_j^*(t)\hat{a}_j\right]. \quad (3.4a)$$

$\hat{S}(\xi(t))$ is the two-mode squeeze operator, defined by

$$\hat{S}(\xi(t)) = \exp\left[\xi^*(t)\hat{a}_1\hat{a}_2 - \xi(t)\hat{a}_1^+\hat{a}_2^+\right] \quad (3.4b)$$

where

$$z_j(t) = r_j(t)\exp(\mathrm{i}\delta_j(t)) \quad (3.5a)$$

$$\xi(t) = s(t)\exp(\mathrm{i}\theta(t)) \quad (3.5b)$$

supposing that $r_j(t), \delta_j(t), s(t)$ and $\theta(t)$ are real functions of time. For brevity, the time-dependent parameters $z_j(t), \xi(t), s(t)$ and $\theta(t)$ are denoted as z_j, ξ, s and θ (when $t = 0$, denoted as z_{j0}, ξ_0, s_0 and θ_0), and may be determined by equations (2.1) and (3.2). By using following relational expressions (omitting their Hermitian conjugate formats),

$$\hat{D}_j(z_j)\hat{a}_j\hat{D}_j^+(z_j) = \hat{a}_j - z_j \quad (3.6a)$$

$$\hat{S}(\xi)\hat{a}_1\hat{S}^+(\xi) = \hat{a}_1\cosh s + \hat{a}_2^+\exp(\mathrm{i}\theta)\sinh s \quad (3.6b)$$

$$\hat{S}(\xi)\hat{a}_2\hat{S}^+(\xi) = \hat{a}_2\cosh s + \hat{a}_1^+\exp(\mathrm{i}\theta)\sinh s \quad (3.6c)$$

and from (3.3) and (3.6), we get

$$\hat{I}(t) = \hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2 - \hat{a}_1^+z_1 - \hat{a}_1z_1^* + \hat{a}_2^+z_2 + \hat{a}_2z_2^* + z_1z_1^* - z_2z_2^*. \quad (3.7)$$

From (2.1) and (3.7), the following commutation relation can be obtained:

$$\begin{aligned}[\hat{I}(t), \hat{H}(t)] = {} & \left\{\left[\Omega_1^0(t)z_1 + G_{12}(t)z_2^*\exp(\mathrm{i}\phi(t)) + G_1(t)\exp(\mathrm{i}\varphi_1(t))\right]\hat{a}_1^+\right.\\ & \left. - \left[\Omega_2^0(t)z_2 + G_{12}(t)z_1^*\exp(\mathrm{i}\phi(t)) + G_2(t)\exp(\mathrm{i}\varphi_2(t))\right]\hat{a}_2^+\right\} - \text{H.c.}\\ & + G_1(t)[z_1\exp(-\mathrm{i}\varphi_1(t)) - z_1^*\exp(\mathrm{i}\varphi_1(t))]\\ & - G_2(t)[z_2\exp(-\mathrm{i}\varphi_2(t)) - z_2^*\exp(\mathrm{i}\varphi_2(t))]. \quad (3.8)\end{aligned}$$

Here H.c. stands for Hermitian conjugate. Combining (3.2), (3.7) and (3.8), we obtain following independent differential equations:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2G_{12}(t)\coth 2s\cos(\theta - \phi(t)) - \Omega_1^0(t) - \Omega_2^0(t) \quad (3.9a)$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = G_{12}(t)\sin(\theta - \phi(t)) \quad (3.9b)$$

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = -\mathrm{i}\left[\Omega_1^0(t)z_1 + G_{12}(t)z_2^*\exp(\mathrm{i}\phi(t)) + G_1(t)\exp(\mathrm{i}\varphi_1(t))\right] \quad (3.10a)$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}t} = -\mathrm{i}\left[\Omega_2^0(t)z_2 + G_{12}(t)z_1^*\exp(\mathrm{i}\phi(t)) + G_2(t)\exp(\mathrm{i}\varphi_2(t))\right]. \quad (3.10b)$$

It is easy to show that the displacement operator parameters z_1 and z_2 depend on the driving part and the two-mode coupled part simultaneously, whereas the two-mode squeeze operator parameters, the squeeze factor s and the squeeze angle θ, are independent of the driving part (see (3.4), (3.5), (3.9) and (3.10)). By substituting the solution z_1 and z_2 from (3.10) into (3.7), we can obtain the time-dependent invariant $\hat{I}(t)$.

Let $|n_1, n_2\rangle$ be the eigenstate vector of the operator $\hat{I}_0$, i.e.

$$\hat{I}_0|n_1, n_2\rangle = (n_1 - n_2)|n_1, n_2\rangle. \tag{3.11}$$

Rewriting $\hat{D}_1(z_1)\hat{D}_2(z_2)S(\xi)|n_1, n_2\rangle$ as $|z_1, z_2, \xi, n_1, n_2\rangle$, i.e.

$$\hat{D}_1(z_1)\hat{D}_2(z_2)S(\xi)|n_1, n_2\rangle = |z_1, z_2, \xi, n_1, n_2\rangle \tag{3.12}$$

we have

$$\begin{aligned}\hat{I}(t)|z_1, z_2, \xi, n_1, n_2\rangle &= \hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)\hat{I}_0\hat{S}^{+}(\xi)\hat{D}_2^{+}(z_2)\hat{D}_1^{+}(z_1)\hat{D}_1(z_1)\hat{D}_2(z_2)S(\xi)|n_1, n_2\rangle \\ &= (n_1 - n_2)|z_1, z_2, \xi, n_1, n_2\rangle.\end{aligned} \tag{3.13}$$

Obviously, the state $\hat{D}_1(z_1)\hat{D}_2(z_2)S(\xi)|n_1, n_2\rangle$ is an eigenstate vector of the operator $\hat{I}(t)$. In addition, by using equation (3.12) and $\sum_{n_1,n_2} |n_1, n_2\rangle\langle n_1, n_2| = 1$, we can prove the completeness relation of $|z_1, z_2, \xi, n_1, n_2\rangle$,

$$\begin{aligned}&\sum_{n_1,n_2} |z_1, z_2, \xi, n_1, n_2\rangle\langle z_1, z_2, \xi, n_1, n_2| \\ &\quad = \hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)\sum_{n_1,n_2} |n_1, n_2\rangle\langle n_1, n_2|\hat{S}^{+}(\xi)\hat{D}_2^{+}(z_2)\hat{D}_1^{+}(z_1) \\ &\quad = \hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)\hat{S}^{+}(\xi)\hat{D}_2^{+}(z_2)\hat{D}_1^{+}(z_1) = 1.\end{aligned} \tag{3.14}$$

According to the Lewis–Riesenfeld quantum-invariant theory [26, 27], the general solution of the time-dependent Schrödinger equation (3.1) can be expressed as

$$\begin{aligned}|\psi(t)\rangle &= \sum_{n_1,n_2} C_{n_1n_2}\exp[\mathrm{i}\alpha_{n_1n_2}(t)]|z_1, z_2, \xi, n_1, n_2\rangle \\ &= \sum_{n_1,n_2} C_{n_1n_2}\exp[\mathrm{i}\alpha_{n_1n_2}(t)]\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)|n_1, n_2\rangle\end{aligned} \tag{3.15}$$

where $\alpha_{n_1n_2}(t)$ is Lewis–Riesenfeld phase, it can be decomposed into a geometric phase $\gamma_{n_1n_2}(t)$ and a dynamical phase $\beta_{n_1n_2}(t)$, namely

$$\alpha_{n_1n_2}(t) = \gamma_{n_1n_2}(t) + \beta_{n_1n_2}(t) \tag{3.16a}$$

where

$$\gamma_{n_1n_2}(t) = \int_0^t \langle z_1, z_2, \xi, n_1, n_2|\mathrm{i}\frac{\partial}{\partial t}|z_1, z_2, \xi, n_1, n_2\rangle\,\mathrm{d}t \tag{3.16b}$$

$$\beta_{n_1n_2}(t) = -\int_0^t \langle z_1, z_2, \xi, n_1, n_2|\hat{H}(t)|z_1, z_2, \xi, n_1, n_2\rangle\,\mathrm{d}t. \tag{3.16c}$$

Obviously, $\alpha_{n_1n_2}(0) = 0$. By adopting the following relations (omitting their Hermitian conjugate formats) [28]

$$\hat{S}^{+}(\xi)\hat{a}_1\hat{S}(\xi) = \hat{a}_1\cosh s - \hat{a}_2^{+}\exp(\mathrm{i}\theta)\sinh s \tag{3.17a}$$

$$\hat{S}^{+}(\xi)\hat{a}_2\hat{S}(\xi) = \hat{a}_2\cosh s - \hat{a}_1^{+}\exp(\mathrm{i}\theta)\sinh s \tag{3.17b}$$

$$\hat{D}_j^{+}(z_j)\hat{a}_j\hat{D}_j(z_j) = \hat{a}_j + z_j \tag{3.17c}$$

$$\hat{D}_j(z_j) = \exp\left(z_j\hat{a}_j^{+}\right)\exp(-z_j^{*}\hat{a}_j)\exp\left(-\tfrac{1}{2}z_j^{*}z_j\right) \tag{3.18a}$$

$$\hat{S}(\xi) = \exp\left(-\hat{a}_1^{+}\hat{a}_2^{+}\,\mathrm{e}^{\mathrm{i}\theta}\tanh s\right)\exp\left[-\left(\hat{a}_1^{+}\hat{a}_1 + \hat{a}_2^{+}\hat{a}_2 + 1\right)\ln(\cosh s)\right]\exp(\hat{a}_1\hat{a}_2\,\mathrm{e}^{-\mathrm{i}\theta}\tanh s) \tag{3.18b}$$

and substituting (2.1), together with (3.12), into (3.16), a rather lengthy calculation yields the results

$$\gamma_{n_1n_2}(t) = \int_0^t \left[\frac{\mathrm{i}}{2}(\dot{z}_1 z_1^* + \dot{z}_2 z_2^* - z_1\dot{z}_1^* - z_2\dot{z}_2^*) - (n_1+n_2+1)\dot{\theta}\sinh^2 s\right]\mathrm{d}t \tag{3.19a}$$

$$\begin{aligned}\beta_{n_1n_2}(t) = -\int_0^t &\left\{\left[\Omega_1^0(t)n_1 + \Omega_2^0(t)n_2\right]\cosh^2 s + \left[\Omega_1^0(t)n_2 + \Omega_2^0(t)n_1 + \Omega_1^0(t) + \Omega_2^0(t)\right]\sinh^2 s\right.\\ &\left. + \Omega_1^0(t)z_1z_1^* + \Omega_2^0(t)z_2z_2^*\right\}\mathrm{d}t - \int_0^t \{G_{12}(t)[z_1^*z_2^*\exp(\mathrm{i}\phi(t)) + z_1z_2\exp(-\mathrm{i}\phi(t))\\ &- (n_1+n_2+1)\cosh(\theta-\phi(t))\sinh 2s] + G_1(t)[z_1^*\exp(\mathrm{i}\varphi_1(t))\\ &+ z_1\exp(-\mathrm{i}\varphi_1(t))] + G_2(t)[z_2^*\exp(\mathrm{i}\varphi_2(t)) + z_2\exp(-\mathrm{i}\varphi_2(t))] + G(t)\}\,\mathrm{d}t.\end{aligned} \tag{3.19b}$$

Here $\dot{z}_1 = \frac{\mathrm{d}z_1}{\mathrm{d}t}, \dot{z}_2 = \frac{\mathrm{d}z_2}{\mathrm{d}t}, \dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$. Substituting (3.19) into (3.16*a*), we may also write the Lewis–Riesenfeld phase as

$$\alpha_{n_1n_2}(t) = -[\varepsilon_1(t)n_1 + \varepsilon_2(t)n_2] + \sigma(t) \tag{3.20}$$

where

$$\varepsilon_1(t) = \int_0^t \left[\Omega_1^0(t) - G_{12}(t)\cos(\theta-\phi(t))\tanh s\right]\mathrm{d}t \tag{3.21a}$$

$$\varepsilon_2(t) = \int_0^t \left[\Omega_2^0(t) - G_{12}(t)\cos(\theta-\phi(t))\tanh s\right]\mathrm{d}t \tag{3.21b}$$

$$\begin{aligned}\sigma(t) = \int_0^t &\left\{G_{12}(t)\cos(\theta-\phi(t))\tanh s - \frac{G_1(t)}{2}[z_1^*\exp(\mathrm{i}\varphi_1(t)) + z_1\exp(-\mathrm{i}\varphi_1(t))]\right.\\ &\left. - \frac{G_2(t)}{2}[z_2^*\exp(\mathrm{i}\varphi_2(t)) + z_2\exp(-\mathrm{i}\varphi_2(t))] - G(t)\right\}\mathrm{d}t.\end{aligned} \tag{3.22}$$

At $t = 0$, the initial state vector of the system is

$$|\psi(0)\rangle = \hat{D}_1(z_{10})\hat{D}_2(z_{20})\hat{S}(\xi_0)\sum_{n_1,n_2} C_{n_1n_2}|n_1, n_2\rangle. \tag{3.23}$$

From (3.20)–(3.23) and (3.15), we obtain the time evolution of the quantum states

$$\begin{aligned}|\psi(t)\rangle = &\exp(\mathrm{i}\sigma(t))\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)\\ &\times\exp\left[-\mathrm{i}\left(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2\right)\right]\hat{S}^+(\xi_0)\hat{D}_2^+(z_{20})\hat{D}_1^+(z_{10})|\psi(0)\rangle.\end{aligned} \tag{3.24}$$

In what follows we suppose $s(t=0) = s_0 = 0$, from (3.4*b*) and (3.5*b*), we have

$$\hat{S}(\xi_0) = \hat{S}[s_0\exp(\mathrm{i}\theta_0)] = \hat{S}^+(\xi_0) = 1 \tag{3.25}$$

the state vector (3.24) becomes

$$\begin{aligned}|\psi(t)\rangle = &\exp(\mathrm{i}\sigma(t))\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)\exp\left[-\mathrm{i}\left(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2\right)\right]\\ &\times\hat{D}_2^+(z_{20})\hat{D}_1^+(z_{10})|\psi(0)\rangle.\end{aligned} \tag{3.26}$$

In special situations, the solution is relatively straightforwardly obtained. For example, if the quantum system starts in a coherent state $|\psi(0)\rangle = \hat{D}_1(z_{10})\hat{D}_2(z_{20})|0, 0\rangle$, the state vector

(3.26) can be rearranged to the following normal ordered form by using (3.18) and the Hermitian conjugate format of (3.6*a*):

$$|\psi(t)\rangle = \exp(\mathrm{i}\sigma(t))\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)|0,0\rangle = \exp(\mathrm{i}\sigma_1)\exp\left[-\exp(\mathrm{i}\theta)\tanh s\hat{a}_1^+\hat{a}_2^+\right] \times \exp\left\{[z_1 + z_2^*\exp(\mathrm{i}\theta)\tanh s]\hat{a}_1^+\right\}\exp\left\{[z_2 + z_1^*\exp(\mathrm{i}\theta)\tanh s]\hat{a}_2^+\right\}|0,0\rangle. \tag{3.27a}$$

Here

$$\exp(\mathrm{i}\sigma_1) = \exp\left[\mathrm{i}\sigma(t) - \ln(\cosh s) - \tfrac{1}{2}(z_1^*z_1 + z_2^*z_2) - z_1^*z_2^*\exp(\mathrm{i}\theta)\tanh s\right]. \tag{3.27b}$$

On the other hand, from (3.26), we can obtain the evolution operator

$$\hat{U}(t,0) = \exp(\mathrm{i}\sigma(t))\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{S}(\xi)\exp\left[-\mathrm{i}\big(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2\big)\right]\hat{D}_2^+(z_{20})\hat{D}_1^+(z_{10}). \tag{3.28}$$

In the Heisenberg picture, the time evolution of arbitrary operator is given by

$$\hat{A}(t) = \hat{U}^+(t,0)\hat{A}(0)\hat{U}(t,0). \tag{3.29}$$

It is useful to obtain the time evolution of the annihilation operator. Combining equations (3.17), (3.6), (3.29) and relations

$$\begin{aligned}\exp\left[\mathrm{i}(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2)\right]\hat{a}_j\exp\left[-\mathrm{i}\big(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2\big)\right] &= \exp(-\mathrm{i}\varepsilon_j(t))\hat{a}_j\\ \exp\left[\mathrm{i}(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2)\right]\hat{a}_j^+\exp\left[-\mathrm{i}\big(\varepsilon_1(t)\hat{a}_1^+\hat{a}_1 + \varepsilon_2(t)\hat{a}_2^+\hat{a}_2\big)\right] &= \exp(\mathrm{i}\varepsilon_j(t))\hat{a}_j^+\end{aligned} \tag{3.30}$$

we have

$$\hat{a}_1(t) = \cosh s\exp(-\mathrm{i}\varepsilon_1)(\hat{a}_1 - z_{10}) - \sinh s\exp[\mathrm{i}(\varepsilon_2 + \theta)]\big(\hat{a}_2^+ + z_{20}^*\big) + z_1 \tag{3.31a}$$

$$\hat{a}_2(t) = \cosh s\exp(-\mathrm{i}\varepsilon_2)(\hat{a}_2 - z_{20}) - \sinh s\exp[\mathrm{i}(\varepsilon_1 + \theta)]\big(\hat{a}_1^+ + z_{10}^*\big) + z_2. \tag{3.31b}$$

The creation operators $\hat{a}_1^+(t)$ and $\hat{a}_2^+(t)$ follow (3.31) by taking the Hermitian conjugate. The time dependence of other combinations of annihilation and creation operators can be obtained in a similar manner.

4. An explicitly analytical solution with specific harmonic time-dependent functions

Having presented in detail a very general solution in the previous section, which can be evaluated by the standard numerical method, we are now in a position to find an analytical solution by choosing proper harmonic time-dependent functions (see (2.3) and (2.4)) as follows.

It is convenient to define the parameters

$$G_{12}(t) = \frac{\kappa\varepsilon}{2} = g, \qquad k = \frac{\Delta\Omega}{2g}. \tag{4.1}$$

From (2.4), (3.9) and (4.1), we have

$$\frac{\mathrm{d}\sin(\theta + 2\Omega t + \Phi)}{\mathrm{d}s} + 2\coth 2s\sin(\theta + 2\Omega t + \Phi) = -2k. \tag{4.2}$$

From (2.3), (2.4), (3.10) and (4.1), we get

$$\frac{\mathrm{d}^2\left[z_1\exp\left(\mathrm{i}\Omega_1^0 t\right)\right]}{\mathrm{d}t^2} + \mathrm{i}2gk\frac{\mathrm{d}\left[z_1\exp\left(\mathrm{i}\Omega_1^0 t\right)\right]}{\mathrm{d}t} - g^2 z_1\exp\left(\mathrm{i}\Omega_1^0 t\right) = \exp\left(\mathrm{i}\Omega_1^0 t\right) \times \left\{\left[\mathrm{i}gG_2\cos\omega_2 t\exp[-\mathrm{i}(2\Omega t + \Phi] + G_1[\mathrm{i}\omega_1\sin\omega_1 t - \left(\Omega_2^0 - 2\Omega\right)\cos\omega_1 t\right]\right\} \tag{4.3a}$$

$$\frac{\mathrm{d}^2\left[z_2\exp\left(\mathrm{i}\Omega_2^0 t\right)\right]}{\mathrm{d}t^2}+\mathrm{i}2gk\frac{\mathrm{d}\left[z_2\exp\left(\mathrm{i}\Omega_2^0 t\right)\right]}{\mathrm{d}t}-g^2z_2\exp\left(\mathrm{i}\Omega_2^0 t\right)=\exp\left(\mathrm{i}\Omega_2^0 t\right)$$
$$\times\left\{\left[\mathrm{i}gG_1\cos\omega_1 t\exp[-\mathrm{i}(2\Omega t+\Phi]+G_2[\mathrm{i}\omega_2\sin\omega_2 t-\left(\Omega_1^0-2\Omega\right)\cos\omega_2 t\right]\right\}. \tag{4.3b}$$

Using the condition $s \geqslant 0$ and the initial condition $s_0 = 0$, we can solve (3.9*b*), (4.2) and (4.3) separately for three different cases: $k^2 < 1$, $k^2 > 1$ and $k^2 = 1$.

Thus, in the case $k^2 < 1$, the solution of (3.9) and (3.10) is

$$s=\ln\left[\frac{\sinh\left(gt\sqrt{1-k^2}\right)+\sqrt{1-k^2+\sinh^2\left(gt\sqrt{1-k^2}\right)}}{\sqrt{1-k^2}}\right] \tag{4.4a}$$

$$\theta=\pi-2\Omega t-\Phi+\arg\left(\cosh\left(gt\sqrt{1-k^2}\right)+\mathrm{i}\frac{k}{\sqrt{1-k^2}}\sinh\left(gt\sqrt{1-k^2}\right)\right) \tag{4.4b}$$

$$\begin{aligned}z_1=z_{1h}&-\frac{\mathrm{i}gG_2\exp(-\mathrm{i}\Phi)}{2}\left[\frac{\exp[-\mathrm{i}(2\Omega-\omega_2)t]}{\left(2\Omega-\Omega_1^0-\omega_2\right)^2-2gk\left(2\Omega-\Omega_1^0-\omega_2\right)+g^2}\right.\\&\left.+\frac{\exp[-\mathrm{i}(2\Omega+\omega_2)t]}{\left(2\Omega-\Omega_1^0+\omega_2\right)^2-2gk\left(2\Omega-\Omega_1^0+\omega_2\right)+g^2}\right]\\&-\frac{G_1}{2}\left[\frac{\left(2\Omega-\Omega_2^0+\omega_1\right)\exp(\mathrm{i}\omega_1 t)}{\left(\Omega_1^0+\omega_1\right)^2+2gk\left(\Omega_1^0+\omega_1\right)+g^2}\right.\\&\left.+\frac{\left(2\Omega-\Omega_2^0-\omega_1\right)\exp(-\mathrm{i}\omega_1 t)}{\left(\Omega_1^0-\omega_1\right)^2+2gk\left(\Omega_1^0-\omega_1\right)+g^2}\right]\end{aligned} \tag{4.5a}$$

$$\begin{aligned}z_2=z_{2h}&-\frac{\mathrm{i}gG_1\exp(-\mathrm{i}\Phi)}{2}\left[\frac{\exp[-\mathrm{i}(2\Omega-\omega_1)t]}{\left(2\Omega-\Omega_2^0-\omega_1\right)^2-2gk\left(2\Omega-\Omega_2^0-\omega_1\right)+g^2}\right.\\&\left.+\frac{\exp[-\mathrm{i}(2\Omega+\omega_1)t]}{\left(2\Omega-\Omega_2^0+\omega_1\right)^2-2gk\left(2\Omega-\Omega_2^0+\omega_1\right)+g^2}\right]\\&-\frac{G_2}{2}\left[\frac{\left(2\Omega-\Omega_1^0+\omega_2\right)\exp(\mathrm{i}\omega_2 t)}{\left(\Omega_2^0+\omega_2\right)^2+2gk\left(\Omega_2^0+\omega_2\right)+g^2}\right.\\&\left.+\frac{\left(2\Omega-\Omega_1^0-\omega_2\right)\exp(-\mathrm{i}\omega_2 t)}{\left(\Omega_2^0-\omega_2\right)^2+2gk\left(\Omega_2^0-\omega_2\right)+g^2}\right]\end{aligned} \tag{4.5b}$$

where the general solution of the homogeneous part of (4.3)

$$z_{jh}=\left[B_j\exp\left(gt\sqrt{1-k^2}\right)+C_j\exp\left(-gt\sqrt{1-k^2}\right)\right]\exp\left[-\mathrm{i}\left(\Omega_j^0+kg\right)t\right] \tag{4.6}$$

and B_j and C_j are integral constants, which are determined by the initial conditions $z_j = z_{j0}$. In the case $k^2 > 1$, the solution of equations (3.9) and (3.10) has the same form as in the case of $k^2 < 1$ with the substitution $\sqrt{1-k^2} \to \mathrm{i}\sqrt{k^2-1}$. In the case $k^2 = 1$, the solution also has the similar form as in the case of $k^2 < 1$; however, the variables s and θ are the limit of (4.4) as $k \to \pm 1$, and z_{jh} in (4.5) must be replaced by $z_{jh} = (B_j + C_j t)\exp\left[-\mathrm{i}\left(\Omega_j^0 + kg\right)t\right]$.

Substituting s, θ, z_1 and z_2 above in various cases into (3.21), (3.22), (3.26) and (3.29), the explicitly analytical formula for the time evolution of the quantum states and the evolution operators of the system can be obtained. In specific cases, our solution can recover to the previous results [8–11]. For example, if the parameters in expressions (2.1*b*), (2.3) and (2.4) $\Omega_1^0 = \Omega_2^0 = G(t) = \omega'$, $G_1 = \lambda$, $G_2 = \sigma$, $\Omega = \omega_0$, $\Phi = \frac{\pi}{2}$ and $g = \omega' - \omega_0$, the Hamiltonian (2.1) can be reduced to the model in [9]. The solution of [9] (i.e. a squeezed coherent state) corresponds to our results in the case of $|\psi(0)\rangle = \hat{D}_1(z_{10})\hat{D}_2(z_{20})|0, 0\rangle$ and $k = -1$ (see (2.5) and (4.1)). This can be done by the following procedure (which is not shown here for brevity), i.e. using the conditions above and supposing that

$$B_1 = \frac{\sigma g}{2}\left[\frac{1}{(\omega_0+\omega_2)^2} + \frac{1}{(\omega_0-\omega_2)^2}\right] - \frac{\lambda g}{2}\left[\frac{1}{(\omega_0+\omega_1)^2} + \frac{1}{(\omega_0-\omega_1)^2}\right], \qquad C_1 = 0,$$

inserting (4.4) and (4.5) into (3.22) and (3.27). If the parameters in the Hamiltonian (2.1) $\Omega_j^0(t) = \Omega_j^0$, $G(t) = 0$, $G_j(t) = 0$ and $G_{12}(t)\exp[\mathrm{i}\phi(t)]$ assume the form of (2.4), and the system starts in a vacuum state $|\psi(0)\rangle = |0, 0\rangle$, i.e. $z_{10} = z_{20} = 0$, by inserting the parameters $s, \theta, z_1 = 0$ and $z_2 = 0$ solved from (4.4) and (4.5) for $k^2 < 1$, into (3.22) and (3.27), we can get expression (A12) in [10], i.e. a detuned two-mode vacuum-squeezed state solved by using Lie group methods. Moreover, if the parameters in the Hamiltonian (2.1) $\Omega_1^0(t) = \omega_a$, $\Omega_2^0(t) = \omega_b$, $G_j(t) = 0$, $G_{12}(t) = g$, $G(t) = (\omega_a + \omega_b)/2$ and $\phi(t) = -\frac{\pi}{2} - 2\omega t$, our model can lead to the results in [8].

5. Summary and conclusions

In the present study we have derived a closed solution of the Schrödinger equation for a generalized non-degenerate optical parametric down-conversion whose Hamiltonian includes an arbitrarily time-dependent driving part and a two-mode coupled part. In the case of the harmonic time-dependent parameters we have further obtained an explicitly analytical solution. It is not difficult to show that the dynamics of a generalized parametric down-conversion directly leads to the production of generalized squeezed states. By choosing different initial states and parameters in the Hamiltonian (2.1), one can obtain various generalized squeezed states. For instance, we may arrive at a detuned two-mode vacuum-squeezed state [10] for the initial vacuum state $|\psi(0)\rangle = |0, 0\rangle$, and a squeezed coherent state [9] for the initial coherent state. Moreover, the choice $|\psi(0)\rangle = |n_1, n_2\rangle$ results in the family of the two-mode squeezed number states, and so forth. It is also shown that the unitary squeeze operator that may generate the two-mode squeezed effect is independent of the driving part, whereas the unitary displacement operator depends on the driving part and the two-mode coupled part simultaneously (see equations (3.9) and (3.10)). With the evolution operator (3.29) it is also straightforward to calculate the time evolution and expectation value of the arbitrary operator; thereby we can further study the various quantum properties of the quantum system, for example, the quantum fluctuation of the two-mode squeezed states, the signal-to-noise ratio of the signal (or idler) mode and so on.

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